# ON THE STABILIZATION OF UNSTABLE MOTIONS BY ADDITIONAL FORCES WHEN THE FEEDBACK LOOP IS INCOMPLETE

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*PMM Vol.27, No.4, 1963, pp.641-663*

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*(Received April 16, 1963)*

This paper deals with the problem of designing the control action which stabilizes the otherwise unstable motion of the controlled object. The problem is complicated by the limitation on the flow of information in the feedback loop. The solution is based upon the theory of stability of motion [1,2], the theory of the analytic design of control systems [3] and the theory of controllability and observability of linear systems [4]. A numerical example will be examined.

**1. Statement of the problem.** We shall consider a controlled object, the state of which is described by its phase coordinates $z_i(t)$ ($t \geqslant 0$, $i = 1, \ldots, n$). Let this object be confirmed by the directing influences $u_j$, ($j = 1, \ldots, r$), subject to the coordinates $z_i$ by the vector differential equation

$$dz/dt = f[t, z, u] \tag{1.1}$$

Here $f$ is a given $n$-dimensional vector function, $z$ is an $n$-dimensional vector of coordinates $\{z_i\}$, $u$ is the $r$-dimensional vector of the control forces $\{u_j\}$.

We shall assume that we examine the motion $z = z^{\circ}(t)$, which follows from (1.1) when $u(t) \equiv 0$ and for some given initial conditions $z^{\circ}(0) = z_0$, i.e. we are given the motion $z = z^{\circ}(t)$ which can accomplish the objective (1.1) in the absence of the control $u$, but which can be subject to the action of other (programmed) forces included implicitly in the function $f$. Let the motion $z = z^0(t)$, described by equation (1.1) for $u \equiv 0$ be unstable in the sense of Liapunov [1, p.20].

The problem consists in the determination of the forces $u_j$ which will stablize the motion $z^0(t)$. Moreover, it is required that the system work according to the feedback principle, i.e. the quantities $u_j$ at each moment of the process must be determined by the continuous state of the object. The problem considered becomes, therefore, a problem of analytical design of a control system [3]. We shall assume, however, that the problem is complicated by the following circumstances. Let us assume that in the control process, we can measure only the quantities $w_1, \ldots, w_l$, related to $z_1, \ldots, z_n$ by the vector relation

$$w = \varphi\,[t, z] \qquad (\varphi = \{\varphi_k\},\ w = \{w_k\},\ z = \{z_i\},\ k = 1, \ldots, l;\ i = 1, \ldots, n) \tag{1.2}$$

which cannot be solved unambiguously with respect to $z$. Therefore the sought control law must relate the values $u_j$ and $w_k$.

Let us define more accurately the formulation of the problem. We shall construct the equations of the perturbed motion [1, p.21] in the neighborhood of the motion $z^0(t)$.

Let $x = z - z^0(t)$. Then

$$\frac{dx}{dt} = p\,[t, x(t), u(t)], \qquad p\,[t, x, u] = f\,[t, x + z^\circ(t), u] - f\,[t, z^\circ(t), 0] \tag{1.3}$$

Condition (1.2) in terms of the variables $x$ and $y = w - \varphi[t, z^0(t)]$ takes the form

$$y = q\,[t, x], \qquad q\,[t, x] = \varphi\,[t, x + z^\circ(t)] - \varphi\,[t, z^\circ(t)] \tag{1.4}$$

If the values of all coordinates $x_i(t)$ $(i = 1, \ldots, n)$ in the control process could be measured and fed to the regulator, then the problem of stabilizing the motion $x = 0$ (i.e. the motion $z^0(t)$) would be for instance formulated as such: find the equation

$$s\,[u^{(m)}(t), \ldots, u(t), t, x(t)] = 0 \tag{1.5}$$

such that the motion $x = 0$, $u^{(m-1)} = \ldots = u = 0$ is asymptotically stable [1, pp.20,85] by virtue of the equations of the perturbed motion (1.3) and (1.5).

The degree $m \geqslant 0$ of equation (1.5) can be given, or it can be determined by the additional conditions of the problem.

In the case considered in this paper, the given statement of the problem cannot be used because of the impossibility of a direct measure of the vector $x(t)$. Therefore we shall look for the possibility of stabilizing the motion $x = 0$ for regulators with delay. We shall limit ourselves to cases in which the regulator is described by an equation of first order ($m = 1$). Let us formulate the problem.

*Problem 1.1.* Find a differential equation with delay

$$du / dt = U\ [t, y\ (t + \vartheta),\ \ u\ (t + \vartheta)]\quad (-\tau \leqslant \vartheta \leqslant 0,\ \tau = \text{const} > 0)\quad (1.6)$$

such that the motion $x = 0$, $u = 0$ is asymptotically stable [5, p.156] by virtue of the equations of perturbed motion (1.3), (1.4) and (1.6).

In equation (1.6) the coordinates of the $r$-dimensional vector $U$, the functionals $U_j[t,\ y(\vartheta),\ u(\vartheta)]$ are defined for the continuous functions $y_k(\vartheta)$ and $u_j(\vartheta)$ ($-\tau \leqslant \vartheta \leqslant 0$, $j = 1, \ldots, r$; $k = 1, \ldots, l$). The constant $\tau > 0$ can be given or it can be specified in the solution of the problem.

*Note 1.1.* Inasmuch as the functionals $U_j$ depend on the vector $u(t + \vartheta)$, it is assumed that the forces $u(t)$ developed by the regulator can be measured. It is also assumed that the values $y(t)$ and $u(t)$ can be put in memory for an interval of time of duration $\tau$.

*1.2.* The expediency of the introduction of the delay in the control law (1.6) is also substantiated by the fact that only in particular cases is the stabilization of an unstable motion $x = 0$ of system (1.3) possible by means of a control law of the form

$$du/ dt = S\ [t, y\ (t), u\ (t)] \qquad (1.7)$$

where the vector $y$ is defined by relation (1.4) which cannot be solved unambiguously with respect to $x$ (see the example at the end of the paper, p. 1000).

*1.3.* Problem 1.1 can also have the additional requirement of minimizing some functional of the disturbed motions $x(t)$, $u(t)$ of system (1.3), (1.4) and (1.6). Then for instance, the following modification of Problem 1.1 might arise.

*Problem 1.2.* Find the differential equation with delay (1.6), such that the motion $x = 0$, $u = 0$ is asymptotically stable by virtue of the equations of the perturbed motion (1.3), (1.4) and (1.6) and moreover, such that for the motions $x(t)$, $u(t)$ of system (1.3), (1.4) and (1.6) the functional

$$J\ [t_0,\ x_0,\ u_0;\ u] = \int_{t_0}^{\infty} \omega\ [t,\ x\ (t),\ u\ (t),\ u^{(1)}\ (t)]\ dt \qquad (1.8)$$

is minimized for all sufficiently small perturbations $x(t_0) = x_0$, $u(t_0) = u_0$ for every $t_0 \geqslant \tau$. Here $\omega[t, x, u, u^{(1)}]$ is a given analytic function of $x$, $u$, $u^{(1)}$ (for every $t \geqslant \tau$), positive definite with respect to $x$, $u$, $u^{(1)}$ [1, p.80]; it is also assumed that the control process with respect to the equations (1.3), (1.4) and (1.6) begins when $t = 0$ ($u^{(1)} = du/dt$).

In the present paper, methods for solving Problems 1.1 and 1.2 are considered, whereupon there is a summing up to some extent of some results pertaining to the theory of the analytic design of regulators and to the similar problem of stabilizability, controllability and observability of the controlled systems [3,4,6-9].

**2. Statement of problem in linear approximation.** Let the vector functions $p[t, x, u]$ and $q[t, x]$ in equations (1.3) and (1.4) be differentiable with respect to $x$ and $u$. Then

$$dx/dt = P(t)\,x + B(t)\,u + \gamma\,[t, x, u], \qquad y = Q(t)\,x + \nu\,[t, x] \quad (2.1)$$

Here $P(t)$ is an $n \times n$-matrix $\{p_{ij}(t)\}$, $B(t)$ is an $n \times r$-matrix $\{b_{ij}(t)\}$, $Q(t)$ an $l \times n$-matrix $\{q_{ij}(t)\}$; the vector function $\gamma[t, x, u]$ and $\nu[t, x]$ have at the point $x = 0$, $u = 0$ for each $t \in [0, \infty)$, an order of smallness greater than that of the quantity

$$\rho = \left[\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{r} u_j^2\right]^{1/2} \to 0 \quad (2.2)$$

We shall assume that for all $t \geqslant 0$ the matrices $P(t)$, $B(t)$ and $Q(t)$ are continuous and uniformly bounded, and we shall also assume that the condition

$$\begin{gathered} |\gamma_i\,[t, x, u]| \leqslant \varepsilon\rho^2, \qquad |\nu_j\,[t, x]| \leqslant \varepsilon\rho^2 \\ (\varepsilon > 0,\ \rho < \delta,\ \delta > 0,\ i = 1, \ldots, n;\ j = 1, \ldots, l) \end{gathered} \quad (2.3)$$

is uniformly satisfied.

The linear approximation for the equations (1.3) and (1.4) has the form

$$dx/dt = P(t)\,x + B(t)\,u \quad (2.4)$$

$$y = Q(t)\,x \quad (2.5)$$

Therefore Problem 1.1 is formulated as such in the linear approximation.

*Problem 2.1.* Find the linear differential equation with delay

$$du/dt = W\,[t, y\,(t+\vartheta), u\,(t+\vartheta)] \quad (-\tau \leqslant \vartheta \leqslant 0) \tag{2.6}$$

such that the motion $x = 0$, $u = 0$ is asymptotically stable by virtue of the equations of perturbed motion (2.4) to (2.6).

Here $W[t, y(\vartheta), u(\vartheta)]$ is an $r$-dimensional vector, the coordinates of which, $W_j[t, y(\vartheta), u(\vartheta)]$, are linear functionals [10, p.165] determined from the continuous functions $y(\vartheta)$, $u(\vartheta)$ $(-\tau \leqslant \vartheta \leqslant 0)$ and depending continuously on $t$ for $0 \leqslant t < \infty$.

Problem 1.2 in the first approximation reduces to the following: let the function $\omega[t, x, u, u^{(1)}]$ have an expansion

$$\omega\,[t, x, u, u^{(1)}] = \omega_{(1)} + \varkappa = \tag{2.7}$$

$$= \sum_{i,j=1}^{n} c_{ij}\,(t)\,x_i x_j + \sum_{i,j=1}^{r} d_{ij}\,(t)\,u_i u_j + \sum_{i,j=1}^{r} d_{ij}\,(t)\,u_i^{(1)} u_j^{(1)} + \varkappa\,[t, x, u, u^{(1)}] \quad \omega_{(1)} + \varkappa$$

where the condition

$$|\varkappa\,[t, x, u, u^{(1)}]| \leqslant \varepsilon \Big(\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{r} u_j^2 + \sum_{j=1}^{r} [u_j^{(1)}]^2\Big)^{1+\alpha} \quad (t \geqslant 0,\ \alpha > 0) \tag{2.8}$$

is satisfied for

$$\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{r} u_j^2 + \sum_{j=1}^{r} (u_j^{(1)})^2 \leqslant \delta^2 \qquad (\delta > 0)$$

In addition to this, it is natural to assume that the forms

$$\sum c_{ij} x_i x_j, \qquad \sum d_{ij} u_i u_j, \qquad \sum e_{ij} u_i^{(1)} u_j^{(1)}$$

are positive definite.

Then we have the problem.

*Problem* 2.2. Find the differential equation (2.6) such that the motion $x = 0$, $u = 0$ is asymptotically stable on the basis of equations (2.4), (2.5) and (2.6), and such that, in such a case, for motions of $x(t)$, $u(t)$ of the system (2.4) to (2.6) the functional is minimum for all $x(t_0) = x_0$, $u(t_0) = u_0$ and $t_0 \geqslant \tau$.

$$J_2\,[t_0, x_0, u_0; u] = \tag{2.9}$$

$$= \int\limits_{t_0} \Big[ \sum_{i,j=1} [c_{ij}(t)\, x_i(t)\, x_j(t)] + \sum_{i,j=1} [d_{ij}(t) u_i(t)\, u_j(t) + e_{ij}(t)\, u_i^{(1)}(t)\, u_j^{(1)}(t)] \Big] dt$$

*Note 2.1.* The assumption of linearity of equation (2.6) made beforehand does not reduce essentially the possibilities of solution of Problem 2.2, since it is known [3] that similar problems of the minimum of a quadratic functional have for solution a linear control law.

**3. Auxiliary definitions and specifications.** Let $C$ be some matrix. We shall designate by $C^*$, $C_{[i]}$, $C^{[i]}$, $C_{ij}$, $C^{-1}$ the transposed matrix of $C$, the $i$th line, the $i$th column, the $(i, j)$th element, and inverse matrix of $C$ if it exists, respectively. By the symbol $X[t\ ,\ t]$ we shall denote the fundamental solution matrix for the equation

$$dx \,/\, dt = P(t)\, x \tag{3.1}$$

Then $X[t_0, t_0] = E$ is the unit matrix. It is known [11, p.171] that equation

$$\frac{d\,(X^{-1})^*}{dt} = -P^*(X^{-1})^* \tag{3.2}$$

is valid.

The solution of equation (2.4) is determined by Cauchy's formula [11, p.172]

$$x(t) = X[t_0, t]\, x(t_0) + \int\limits_{t_0}^{t} X[t_0, t]\, X^{-1}[t_0, \vartheta]\, B(\vartheta)\, u(\vartheta)\, d\vartheta \tag{3.3}$$

Let $\tau$ be some positive number. We shall denote by $H[t, \tau, \vartheta]$ the following $n \times r$-matrix

$$H[t, \tau, \vartheta] = X[t, t+\tau]\, X^{-1}[t, t+\vartheta]\, B(t+\vartheta) \quad (t \geqslant 0,\ 0 \leqslant \vartheta \leqslant \tau) \tag{3.4}$$

We shall denote by the symbol $\|c\|_\tau$ the norm of the vector-function $\{c_i(\vartheta)\}$ $(i = 1, \ldots, k,\ 0 \leqslant \vartheta \leqslant \tau)$

$$\|c\|_\tau = \Big( \int\limits_0^\tau \Big[ \sum_{i=1}^{k} c_i^2(\vartheta) \Big] d\vartheta \Big)^{1/2} \tag{3.5}$$

*Definition 3.1.* Equation (2.4) satisfies the condition $(3.1, \tau)$ if the quadratic form of the variables $\lambda_i (i = 1, \ldots, n)$

$$\Big\| \sum_{i=1}^{n} \lambda_i H_{[i]}[t, \tau, \vartheta] \Big\|_\tau^2 \tag{3.6}$$

is positive definite (with respect to $t$).

Condition $(3.1, \tau)$ plays an important role in the theory of the controllability of linear systems [4]. This condition means that for every $t$, the rows $H_{[i]}$ of the matrix $H[t, \tau, \vartheta]$ are linearly independent vector functions of $\vartheta$ $(-\tau \leqslant \vartheta \leqslant 0)$ and this linear independence in some sense is uniform with respect to $t$. We shall call condition (3.1, $\tau$) strong if it includes, besides the Definition 3.1, the condition that for every $j$ and $t \geqslant 0$ the inequality

$$\sum_{i=1}^{n} \lambda_i h_{ij} [t, \tau, \vartheta] \neq 0 \quad \text{for } \sum_{i=1}^{n} \lambda_i^2 \neq 0 \text{ almost everywhere for } 0 \leqslant \vartheta \leqslant \tau \qquad (3.7)$$

is satisfied.

This last condition was met in control problems [12-15].

Let us consider the auxiliary system of equations

$$dx / dt = P(t) x + B(t) u, \qquad du / dt = \zeta \qquad (3.8)$$

Let $Z[t_0, t]$ be the fundamental solution matrix of the system

$$dx / dt = P(t) x + B(t) u, \qquad du / dt = 0 \qquad (3.9)$$

We shall form for system (3.8) the matrix $F[t, \tau, \vartheta]$, analogous to the matrix $H$, i.e. let

$$F[t, \tau, \vartheta] = Z[t, t+\tau] Z^{-1} [t, t+\vartheta] \begin{Vmatrix} 0 & 0 \\ 0 & E_r \end{Vmatrix} \qquad (3.10)$$

where $E_r$ is a unit $r \times r$-matrix.

The following statement, given without proof, is valid.

*Lemma 3.1.* Let the elements $b_{ij}(t)$ of the matrix $B(t)$ have for $t \geqslant 0$ continuous and uniformly bounded derivatives $b_{ij}^{(1)}(t)$. Then the fulfillment of condition (3.1, $\tau$) for system (3.8) follows from the fulfillment of this condition for equation (2.4). In other words, from the positive-definiteness of the form (3.6) follows the positive-definiteness of the form

$$\left\| \sum_{i=1}^{n+r} \lambda_i F_{[i]} [t, \tau, \vartheta] \right\|_\tau^2 \qquad (3.11)$$

uniformly with respect to $t \geqslant 0$.

*Note 3.1.* If the requirement of uniform boundedness of the derivatives $b_{ij}^{(1)}(t)$ is not satisfied, then the fulfillment of (3.1, $\tau$) for (3.8) may not follow from the fulfillment of (3.1, $\tau$) for (2.4). For instance the scalar equation

$$dx / dt = b(t)\, u, \qquad b(t) = \sin kt \quad \text{for } 2(k-1)\pi \leqslant t \leqslant 2k\pi \tag{3.12}$$

satisfies the condition (3.1, $\tau$) for $\tau = 2\pi$. However, the system

$$dx / dt = b(t)\, u, \qquad du / dt = \zeta$$

does not satisfy condition (3.1, $\tau$) for $\tau = 2\pi$.

We shall present sufficient conditions which guarantee the fulfillment of condition (3.1, $\tau$). Let the functions $p_{ij}(t)$ and $b_{ij}(t)$ have continuous and uniformly bounded derivatives up to the $n$th order, inclusively. We shall examine the sequence of matrices $L_i(t)$ ($i = 1, \ldots, n$) defined by the recurrent relation

$$L_1(t) = B(t), \qquad L_i(t) = \frac{dL_{i-1}}{dt} - P(t) L_{i-1}(t) \tag{3.13}$$

We shall represent by the symbol $l[j, t]$ the $n$-dimensional vector, appearing as the $j$th column of the matrix and we shall examine the quadratic form

$$\sum_{i,k=1}^{n} (l[j_k, t] \cdot l[j_i, t]) \lambda_j \lambda_k \tag{3.15}$$

where the symbol $(l[j_k, t] \times l[j_i, t])$ represents the scalar product of the corresponding vectors.

*Definition 3.2.* Equation (2.4) satisfies the condition (3.2, $\tau$) if on each interval $t_0 < t < t_0 + \tau$ ($t_0 \geqslant 0$) at least one point $t = t^*$ can be found, for which there exists a set of numbers $j_k$($k = 1, \ldots, n$), ($1 \leqslant j_k \leqslant n \times r$) satisfying the condition

$$\sum_{i,k=1}^{n} (l[j_k, t^*] \cdot l[j_i, t^*]) \lambda_i \lambda_k \geqslant \mu \sum_{i=1}^{n} \lambda_i^2 \tag{3.16}$$

where the value $\mu > 0$ does not depend on $t_0$ and $t^*$.

The following statement which follows from known results of the theory of the controllability of linear systems [4] is valid.

*Lemma 3.2.* The fulfillment of (3.2, $\tau$) for equation (2.4) is a sufficient condition for the fulfillment of (3.1, $\tau$) for the same equation.

*Note 3.2.* The set of numbers $j_k$ can depend on $t_0$ and $t^*$. If the inequality (3.16) is satisfied in the strong sense, i.e. for all $t^* \geqslant 0$ and for each set of numbers $j_{i+rk}$ ($k = 0, \ldots, n-1$; $i = 1, \ldots, r$), then (3.1, τ) is satisfied in the strong sense for equation (2.4) (see p. 976). A condition analogous to condition (3.2, τ) was introduced in [12] for the stationary case, in the form of the independence of the vectors $B^{[j]}, \ldots, P^{n-1}B^{[j]}$ as a "common state" condition when the optimal time response problem was studied.

In [4], the common state conditions were considered as conditions of controllability of linear systems. In [16] conditions similar to (3.2, τ) were used for the study of the local controllability of nonlinear systems. In [17] the strong condition (3.2, τ) was used as a sufficient condition of controllability and of continuous dependence of the parameters of the time optimal control in nonstationary linear systems. Conditions of type (3.2, τ) and strong conditions of type (3.2, τ) play an important role in the theory of optimal control (see [13] and other works on optimal control).

We shall also mention [8,9,18-22], where the strong conditions (3.2, τ) were used for the solution of problems of analytical design of control systems with first order approximation, for the solution of problems of stabilizability of stochastic systems, in the study of problems of controllability and of fast response of nonlinear systems in linear approximation, in the investigation of the discontinuous character of the control for optimal time response in nonlinear systems, and in the problem of stochastic pursuit.

We shall give a few more definitions and notations which will be necessary further on when the problem of prediction of the controlled object will be used (see p. 986).

Let $\tau$ be a positive constant. We shall examine the $l \times n$-matrix $G[t, \tau, \vartheta]$ expressed in the following manner with the matrix $Q(t)$ (2.5) and the fundamental matrix $X[t_0, t]$ of equation (3.1)

$$G\,[t, \tau, \vartheta] = Q\,[t + \vartheta]\, X\,[t, t + \vartheta] \qquad (t \geqslant \tau, \quad -\tau \leqslant \vartheta \leqslant 0) \tag{3.17}$$

We shall denote by the symbol $\|c\|_{-\tau}$ the norm of the vector function $\{c_i(\vartheta)\}$ $(i = 1, \ldots, k)$ $(-\tau \leqslant \vartheta \leqslant 0)$

$$\|c\|_{-\tau} = \Big(\int_{-\tau}^{0} \Big[\sum_{i=1}^{k} c_i^2(\vartheta)\Big]\, d\vartheta\Big)^{1/2} \tag{3.18}$$

*Definition 3.3.* Equations (3.1) and (2.5) satisfy condition (3.3, τ) if the quadratic form of the variables $\lambda_i$ $(i = 1, \ldots, n)$

$$\Big\| \sum_{i=1}^{n} \lambda_i G^{[i]}\,[t, \tau, \vartheta] \Big\|_{-\tau}^{2} \tag{3.19}$$

is positive-definite (with respect to $t$).

Condition (3.3, $\tau$) plays an important role in the prediction theory for linear systems [4].* Condition (3.3, $\tau$), according to this theory, appears as a duplication of the condition of controllability (3.1, $\tau$) (see p. 976).

We shall call strong condition (3.3, $\tau$) that condition which includes in addition to Definition 3.3, the requirement that for every $j$ and for $t \geqslant \tau$ the inequality

$$\sum_{i=1}^{n} \lambda_i g_{ji} [t, \tau \ \vartheta] \neq 0 \quad \text{when} \quad \sum_{i=1}^{n} \lambda_i^2 \neq 0 \tag{3.20}$$

is satisfied almost everywhere for $-\tau \leqslant \vartheta \leqslant 0$.

We shall establish the sufficient conditions which guarantee the fulfillment of condition (3.3, $\tau$).

Let us assume that the functions $p_{ij}(t)$ and $q_{ij}(t)$ have continuous and uniformly bounded derivatives up to the $n$th order, inclusively. We shall examine the sequence of matrices $R_i(t)$ ($i = 1, \ldots, n$) defined by the recurrent relation

$$R_1(t) = Q(t), \qquad R_i(t) = \frac{dR_{i-1}}{dt} + R_{i-1}(t) P(t) \tag{3.21}$$

Let $r[j, t]$ be the $n$-dimensional vector representing the $j$th line of the matrix

$$R(t) = \begin{Vmatrix} R_1(t) \\ \vdots \\ R_n(t) \end{Vmatrix} \tag{3.22}$$

and let us examine the quadratic form

$$\sum_{i, k=1}^{n} (r [j_k, t] \cdot r [j_i, t]) \lambda_i \lambda_k \tag{3.23}$$

*Definition 3.4*. Equations (3.1) and (2.5) satisfy the condition (3.4, $\tau$) if on each interval $t_0 - \tau < t < t_0$ ($t_0 \geqslant \tau$) at least one point $t = t^*$ can be found such that there is a set of numbers $j_k$ ($k = 1, \ldots, n$) $1 \leqslant j_k \leqslant n \times l$, which satisfies the condition

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* Kalman, R.E., New methods and Results in Linear Prediction and Filtering Theory. *RIAS. Technical Report*, 1, 1961.

$$\sum_{i,k=1}^{n} (r\,[j_k, t^*]\cdot r\,[j_i, t^*])\,\lambda_i\lambda_k \geqslant \mu \sum_{i=1}^{n} \lambda_i^2 \qquad (\mu > 0) \tag{3.24}$$

The following statement, which follows from the results of the prediction theory for linear systems [4] and appears as a duplication of Lemma 3.2, is valid.

*Lemma 3.3.* Condition (3.4, $\tau$), satisfied for equations (3.1) and (2.5), is a sufficient condition for the fulfillment of (3.3, $\tau$) for these equations.

*Note 3.3.* In this section references were made to some works related to the controllability and prediction theory for linear systems, and having a direct relation to the problems considered in this paper. The problems of controllability and prediction have, however, an extremely large bibliography which was not mentioned in the survey made above. We shall mention in connection with this [23,24], where efficient methods for the solution of problems of control and prediction are described, and also [25], where the problem of stabilization of mechanical systems by dissipative forces is considered. The character of the present paper does not, however, assume a sufficiently complete survey of the pertinent literature.

**4. Solution of the problem in linear approximation.** We shall examine Problems 2.1 and 2.2. The following statement is valid.

*Theorem 4.1.* If conditions (3.2, $\tau$) and (3.4, $\tau$) are satisfied for equations (2.4), (2.5) and (3.1), Problems 2.1 and 2.2 have a solution. The sought control law (2.6) has the form

$$\frac{du}{dt} = A\,(t)\,u\,(t) + \int_{-\tau}^{0} [N\,(t, \vartheta)\,y\,(t+\vartheta) + M\,(t, \vartheta)\,u\,(t+\vartheta)]\,d\vartheta \tag{4.1}$$

where the matrix functions $A$, $N$ and $M$ are continuous with respect to their arguments and are uniformly bounded with respect to $t$. Then the motion $x = 0$, $u = 0$ will by asymptotically stable [5, pp.174,191], on the basis of the equations of the perturbed motion (2.4), (2.5) and (4.1) depending upon the moment $t_0$ of the initial disturbance and upon the initial displacements $x^{\circ}(t_0 + \vartheta)$, $u^{\circ}(t_0 + \vartheta)$ $(-\tau \leqslant \vartheta \leqslant 0)$, i.e. the inequality

$$\sum_{i=1}^{n} x_i^2\,(t) + \sum_{j=1}^{r} u_j^2\,(t) \leqslant \beta e^{-\alpha\,(t-t_0)} \sup_{\vartheta} \Big[\sum_{i=1}^{n} x_i^2\,(t_0+\vartheta) + \sum_{j=1}^{r} u_j^2\,(t_0+\vartheta)\Big] \tag{4.2}$$

$$(t \geqslant t_0,\ \alpha > 0,\ \beta = \mathrm{const} > 0)$$

will be satisfied.

The calculation of the elements $a_{ij}(t)$, $n_{ij}(t, \vartheta)$ and $m_{ij}(t, \vartheta)$ of the matrices $A$, $N$ and $M$ with the required precision reduces to the determination of the fundamental matrix $X[t_0, t]$ of equation (3.1), to the solution of systems of linear algebraic equations and to the solution of Cauchy's problem for a system of ordinary differential equations for some known initial conditions.

*Note 4.1.* From the conditions of Problems 2.1 and 2.2, we can limit ourselves to the determination of equation (4.1) only for $t \geqslant \tau$. Thus we shall solve these problems assuming that in fact the control $u(t)$ which enters equation (4.1), was working in Problem 2.2 for $0 \leqslant t \leqslant t_0 = \tau$, and that in Problem (2.1) the functions $u(t_0 + \vartheta)$, $x(t_0 + \vartheta)$ can be considered as given arbitrarily and independently ($t_0 \geqslant \tau$, $-\tau \leqslant \vartheta \leqslant 0$). In case of necessity, it can be assumed that equation (4.1) found for $t \geqslant \tau$, is extended continuously and in any possible manner until $t = 0$, and it can be assumed that the initial disturbance $x(\vartheta)$, $u(\vartheta)$ is given independently ($-\tau \leqslant \vartheta \leqslant 0$). Problem 2.1 has many solutions; its solution in the form of equation (4.1) is obtained from the solution of Problem 2.2, and gives in its calculation, apparently, a minimum of computing difficulties.

We shall present a discussion, proving the validity of Theorem 4.1, and showing the method of calculation of the elements $a_{ij}$, $n_{ij}$ and $m_{ij}$ of equation (4.1). In agreement with Note 4.1, we shall begin by solving Problem 2.2.

We shall examine the auxiliary problem of the analytical design of an optimal control system.

*Problem 4.1.* Find the control law $\zeta = \zeta^{\circ}[t, x, u]$ which provides an asymptotic stability for the motion $x = 0$, $u = 0$ on the basis of the equations of disturbed motion

$$dx/dt = P(t)\,x(t) + B(t)\,u(t), \qquad du/dt = \zeta \tag{4.3}$$

and such that the control $\zeta = \zeta^{\circ}[t, x, u]$ gives the minimum value of the functional

$$J_2[t_0, x_0, u_0; \zeta] = \tag{4.4}$$

$$= \int_{t_0}^{\infty} \Big[ \sum_{i,j=1}^{n} [c_{ij}(t)\,x_i(t)\,x_j(t)] + \sum_{i,j=1}^{r} [d_{ij}(t)\,u_i(t)\,u_j(t) + e_{ij}(t)\,\zeta_i(t)\,\zeta_j(t)] \Big] dt$$

for all initial conditions $t_0 \geqslant 0$, $x(t_0) = x_0$, $u(t_0) = u_0$ in the class $\Xi$ of continuous admissible controls $\zeta = \zeta[t, x, u]$.

In order to solve Problem 4.1, it is sufficient [3,6,8,26] to find some functions $v^{\circ}(t, x, u)$ and $\zeta^{\circ}(t, x, u)$ satisfying the following conditions:

1. The function $v^{\circ}(t, x, u)$, positive-definite with respect to $x$ and $u$ has an upper bound and increases uniformly indefinitely [5, p.36] when $(x, u) \to \infty$.

2. The derivative $(dv^{\circ}/dt;\ (4.3),\ \zeta)$ of the function $v^{\circ}$ for a fixed control $\zeta$ along the motions $x(t)$, $u(t)$ satisfies the condition corresponding here to the optimal principle [27, p.177]

$$\left(\frac{dv^\circ}{dt};\ (4.3),\ \zeta^\circ\right)=\sum_{i,\,j=1}^{n} c_{ij}(t)\,x_ix_j+\sum_{i,\,j=1}^{r}[d_{ij}(t)\,u_ju_i+e_{ij}(t)\,\zeta_i^\circ\zeta_j^\circ]= \tag{4.5}$$

$$=\min_\zeta\left\{\left(\frac{dv^\circ}{dt};\ (4.3),\ \zeta\right)+\sum_{i,j=1}^{n} c_{ij}(t)\,x_ix_j+\sum_{i,\,j=1}^{r}[d_{ij}(t)\,u_iu_j+e_{ij}(t)\,\zeta_i\zeta_j]\right\}=0$$

(for $\zeta \in \Xi$ for all $x$, $u$, $t \geqslant 0$).

Then $\zeta^{\circ}$ is the optimal control and equation

$$J_2\,[t_0,\,x_0,\,u_0;\,\zeta^\circ]=v^\circ\,(t_0,\,x_0,\,u_0) \tag{4.6}$$

is valid.

The function $v^{\circ}(t, x, u)$ must be sought in the form of a quadratic form of the variables $x_i$, $u_k (i = 1, \ldots, n;\ k = 1, \ldots, r)$

$$v^\circ\,(t,\,x,\,u)=v_2^\circ\,(t,\,x,\,u) \tag{4.7}$$

where the coefficients $\alpha_{js}(t)$ $(j = 1, \ldots, n + r;\ s = 1, \ldots, n + r)$ are time dependent. From equation (4.5) follow the equations [3,4,22] for the coefficients $\alpha_{js}(t)$

$$\frac{d\alpha_{js}\,(t)}{dt}=\xi_{js}\,[t,\,\{\alpha_{\mu,\nu}\,(t)\}] \tag{4.8}$$

where the $\xi_{js}$ are polynomials of second order in $\alpha_{\mu\nu}$.

As a consequence of Condition 1, one should seek the solutions $\alpha_{js}$ of equations (4.8), bounded uniformly for $t \geqslant 0$ and such that the form $v_2^{\circ}(t, x, u)$ (4.7) of $x$ and $u$ is positive-definite with respect to $t$.

The statement deduced from the results mentioned above (see Section 3) is valid.

*Lemma 4.1.* If equation (2.4) satisfies condition (3.2, $\tau$) Problem 4.1 has a solution. Then the optimal control $\zeta^{\circ}$ has the form

$$\zeta_k^\circ [t, x, u] = \sum_{i=1}^{n} \beta_{ki}(t) x_i + \sum_{j=1}^{r} \gamma_{kj}(t) u_j \qquad (k = 1, \ldots, r) \tag{4.9}$$

where the functions $\beta_{ki}(t)$ and $\gamma_{kj}(t)$ are continuous and uniformly bounded with respect to $t \geqslant 0$.

We shall give the proof of Lemma 4.1. According to Lemma 3.2, the realization of condition (3.1, $\tau$) for equation (2.4) follows from the realization of condition (3.2, $\tau$) for the same equation. According to Lemma 3.1, condition (3.1, $\tau$) for system (4.3) follows from (3.1, $\tau$) for (2.4). Thus, system (4.3) satisfies (3.1, $\tau$). According to the theorem on the controllability [4] and the estimates [21] this means that system (4.3) is uniformly controllable by actions $\zeta$ on each interval $\tau$, i.e. for any initial condition

$$t_0 \geqslant 0, \quad x(t_0) = x_0, \quad u(t_0) = u_0, \quad \sum_{i=1}^{n} x_{i0}^2 + \sum_{j=1}^{r} u_{j0}^2 \leqslant 1 \tag{4.10}$$

it is possible to give a control $\zeta^*(t)$ ($t_0 \leqslant t \leqslant t_0 + \tau$) which brings system (4.3) to the state $x = u = 0$ at the instant $t = t_0 + \tau$. Thus the quantity

$$J_2^\tau [t_0, x_0, u_0, \zeta^*] = \tag{4.11}$$
$$= \int_{t_0}^{t_0+\tau} \left\{ \left[ \sum_{i,\, j=1}^{n} c_{ij}(t) x_i(t) x_j(t) \right] + \sum_{i,\, j=1}^{r} [d_{ij}(t) u_i(t) u_j(t) + e_{ij}(t) \zeta_i^*(t) \zeta_j^*(t)] \right\} dt$$

is uniformly bounded with respect to $t_0 \geqslant 0$. It is deducted from there [22, p.228; 28, p.39], that there are solutions $\{\alpha_{js}^T(t)\}$ of equation (4.8) bounded on each interval $0 \leqslant t \leqslant T$ and satisfying the initial condition

$$\alpha_{js}^T(T) = 0 \qquad (j, s = 1, \ldots, n + r) \tag{4.12}$$

From the uniform boundedness of $\alpha_{js}^T(t)$ ($0 \leqslant t \leqslant T$, $T < \infty$) resulting from the boundedness of the quantity (4.11), it can be established by a limiting process that the sought solution $\{\alpha_{js}(t)\}$ of equations (4.8) which guarantees the fulfillment of Condition 1 for $v_2^\circ$, exists and is determined by the equality

$$\alpha_{js}(t) = \lim \alpha_{js}^T(t) \quad \text{as} \quad T \to \infty \tag{4.13}$$

Similar limiting processes are considered in the framework of the problem of the stability of Ricatti's equation [4]. In the usual case, the limiting process (4.13) is considered in detail [29] in which the method of solving the problems of analytical design of control systems on electronic models or on digital computers is also evolved on the same basis (see the example at the end of the paper)*.

Thus, under the conditions of Lemma 4.1, it is possible to find functions $v^\circ = v_2^\circ$ and $\zeta = \zeta^\circ$ satisfying equation (4.5). The function $\zeta^\circ[t, x, u]$ is determined after the determination of $v_2^\circ$ from the equation

$$\frac{\partial}{\partial \zeta}\left\{\left(\frac{dv_2^0}{dt}\ ;\quad (4.3), \zeta\right) + \sum_{i,j=1}^{r} e_{ij}(t)\, \zeta_i \zeta_j\right\} = 0 \tag{4.14}$$

From (4.14) and from the properties of $\alpha_{js}(t)$ established earlier, there follows that the control $\zeta^\circ$ has indeed the form (4.9). Finally, the uniform asymptotic stability of the linear optimal system (4.3)

$$dx\,/\,dt = P(t)\,x + B(t)\,u, \qquad du\,/\,dt = \zeta^\circ\,[t, x, u] \tag{4.15}$$

i.e. the fulfillment of the inequality

$$\sum_{i=1}^{r} x_i^2(t) + \sum_{j=1}^{r} u_j^2(t) \leqslant \left(\sum_{i=1}^{n} x_i^2(t_0) + \sum_{j=1}^{r} u_j^2(t_0)\right)\beta e^{-\alpha(t-t_0)} \tag{4.16}$$
$$(\alpha > 0, \beta > 0 - \text{const},\ t \geqslant t_0)$$

follows from the remark that system (4.15) has [30, p.310] a positive-definite Liapunov function $v = v_2^\circ$ having by virtue of that system a negative-definite derivative

$$\left(\frac{dv_2^\circ}{dt}\ ;\ (4.3), \zeta\right) = -\left[\sum_{i,j=1}^{n} c_{ij}(t)\, x_i x_j + \sum_{i,j=1}^{r} d_{ij}(t)\, u_i u_j + \sum_{i,j=1}^{r} e_{ij}(t)\, \zeta^\circ_i \zeta_j^\circ\right]$$

Thus the validity of Lemma 4.1 is verified.

We shall examine now the auxiliary problem of observation.

*Problem 4.2.* Find a linear operator $Y_{(1)}[t, y(\vartheta), u(\vartheta)]$ defined for $t \geqslant \tau$ for continuous vector functions $\{y_j(\vartheta)\}$ $(j = 1, \ldots, l)$, $\{u_s(\vartheta)\}$ $(s = 1, \ldots, r)$ and satisfying for the solutions $x(t)$, $u(t)$ of equations (2.4), the condition

$$x(t) = Y_{(1)}\,[t, y(t + \vartheta), u(t + \vartheta)] \tag{4.17}$$

* Let us point out that for periodic $P(t)$, $B(t)$, and $\omega_{(1)}$ the functions $\alpha_{js}(t)$ become periodic.

The following statement is valid.

*Lemma 4.2.* Let system (3.1) and (2.6) satisfy condition (3.4, $\tau$); then Problem 4.2 has a solution, and in (4.17) the operator $Y_{(1)}$ can be constructed as

$$Y_{(1)}[t, y(\vartheta), u(\vartheta)] = \int_{-\tau}^{0} \{[Y(t, \vartheta)\, y(\vartheta) + K(t, \vartheta)\, u(\vartheta)\}\, d\vartheta \qquad (4.18)$$

where, consequently, the $n \times l$- and $n \times r$-matrices $Y$ and $K$ have elements $y_{ij}(t, \vartheta)$, $k_{ij}(t, \vartheta)$ continuous and bounded for $t \geqslant \tau$.

The validity of Lemma 4.2 is found from the results of the general prediction theory for linear systems [4]. We shall give the proof of lemma. Let us consider first one more auxiliary problem.

*Problem 4.3.* Find the $n \times l$-matrix $V(t, \vartheta)$ defined for $t \geqslant \tau$ and satisfying the condition

$$x_0 = \int_{-\tau}^{0} V(t, \vartheta)\, G\,[t, \tau, \vartheta]\, x^0 d\vartheta \qquad (4.19)$$

where the matrix $G$ is defined by the equality (3.17) and $x_0$ is an arbitrary $n$-dimensional vector.

In presence of conditions (3.3, $\tau$) Problem (4.3) has a solution [4], and consequently, according to Lemma 3.3, this problem has also a solution when conditions (3.4, $\tau$) are satisfied. The matrix $V(t, \vartheta)$ can be sought in the form

$$V_{[i]}(t, \vartheta) = \lambda^*(i)\, G^*[t, \tau, \vartheta] \qquad (i = 1, \ldots, n) \qquad (4.20)$$

where $\lambda^*(i)$ is a constant $n$-dimensional row vector. From (4.19) and (4.20) follows the equation for $\lambda^*(i)$

$$\delta_{ij} = \lambda^*(i) \int_{-\pi}^{0} (G^*[t, \tau, \vartheta]\, G\,[t, \tau, \vartheta])^{[j]} d\vartheta \qquad (4.21)$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$, $i = 1, \ldots, n$; $j = 1, \ldots, n$. The determinant $\Delta(t)$ of the $n \times n$-matrix

$$\int_{-\tau}^{0} G^* \cdot G\, d\vartheta$$

for conditions (3.3, $\tau$) and uniformly for $t \geqslant \tau$ satisfies the inequality

$$|\Delta(t)| \geqslant \varepsilon \qquad (\varepsilon > 0 = \text{const}) \qquad (4.22)$$

There follows that equation (4.24) can be solved and that the sought matrix $V(t, \vartheta)$ determined by the equality (4.20) will have continuous and uniformly bounded elements $v_{ij}(t, \vartheta)$.

The solution $x(t)$ of equation (2.5) has the form (3.3). From (3.3) we obtain the equality

$$\int_{-\tau}^{0} V(t, \vartheta)\, Q[t+\vartheta]\, x(t+\vartheta)\, d\vartheta = \int_{-\tau}^{0} V(t, \vartheta)\, Q[t+\vartheta]\, X[t+\vartheta]\, x(t)\, d\vartheta +$$
$$+ \int_{-\tau}^{0} \{V(t, \vartheta)\, Q[t+\vartheta] \int_{0}^{\vartheta} X[t, t+\vartheta]\, X^{-1}[t, t+\eta]\, B(t+\eta)\, u(t+\eta)\, d\eta\}\, d\vartheta$$

or from (2.5) and (4.19)

$$\int_{-\tau}^{0} V(t, \vartheta)\, y(t+\vartheta)\, d\vartheta = \tag{4.23}$$
$$= x(t) + \int_{-\tau}^{0} \left\{\int_{0}^{\vartheta} V(t, \vartheta)\, Q[t+\vartheta]\, X[t, t+\vartheta]\, X^{-1}[t, t+\eta]\, B(t+\eta)\, u(t+\eta)\, d\eta\right\} d\vartheta$$

Changing the order of integrations on the right-hand side of (4.23) and replacing $\eta$ by $\vartheta$, we obtain

$$x(t) = \int_{-\tau}^{0} V(t, \vartheta)\, y(t+\vartheta)\, d\vartheta + \int_{-\tau}^{0} \left\{\int_{-\tau}^{\vartheta} V(t, \eta)\, Q(t+\eta)\, X[t, t+\eta]\, d\eta\right\} \times$$
$$\times X^{-1}[t, t+\vartheta]\, B(t+\vartheta)\, u(t+\vartheta)\, d\vartheta$$

If we write

$$\mathrm{Y}(t, \vartheta) = V(t, \vartheta) \tag{4.24}$$
$$K(t, \vartheta) = \left\{\int_{-\tau}^{\vartheta} V(t, \eta)\, Q(t+\eta)\, X[t, t+\eta]\, d\eta\right\} X^{-1}[t, t+\vartheta]\, B(t+\vartheta)$$

we obtain the operator $Y_{(1)}$ (4.18) which satisfies the requirement of Lemma 4.2. This verifies the validity of the lemma.

*Note 4.2.* The solution of Problem 4.2 is not unique. The solution described above, resulting from [4] and leading to the solution of the linear equation (4.21) gives, apparently, some small computation difficulties. However, from the conditions of the problem, it might appear to be expedient to seek the matrix function $V(t, \vartheta)$, determining the operator $Y_{(1)}$ by the equalities (4.24), in the form of discontinuous functions, i.e. as a matrix with piecewise constant elements $v_{ij}(t, \vartheta)$, or as an impulse-matrix. This might lead to a simpler synthesis of the system. Such solutions can be obtained if, for instance, the results of [21] are used in the solution of Problem 4.3. We shall examine briefly the case of $l = 1$ in order to simplify the calculations. We shall first seek the matrix $V(t, \vartheta)$, i.e. in the present case (for $l = 1$) an $n$-dimensional column vector $V^{[1]}(t, \vartheta) = \{v(t, \vartheta)\}$ in the discontinuous form. The matrix $G[t, \tau, \vartheta]$ is for $l = 1$ a row vector $G_{[1]}[t, \tau, \vartheta] = \{g_i(t, \vartheta)\}$ $(i = 1, \ldots, n)$. According to Problem 4.3, it is necessary, thus, to find the vector $\{v_i(t, \vartheta)\}$ having a discontinuous-type character with respect to $\vartheta$ for $-\tau \leqslant \vartheta \leqslant 0$ and satisfying equation (4.19), i.e. here, the equation

$$\left\| \begin{matrix} x_{10} \\ \vdots \\ x_{n0} \end{matrix} \right\| = \int_{-\tau}^{0} \left\| \begin{matrix} g_1 v_1 & \cdots & g_n v_1 \\ \cdots & \cdots & \cdots \\ g_1 v_n & \cdots & g_n v_n \end{matrix} \right\| \cdot \left\| \begin{matrix} x_{10} \\ \vdots \\ x_{n0} \end{matrix} \right\| d\vartheta \tag{4.25}$$

From (4.25) we obtain $n$ systems of linear equations

$$\delta_{ij} = \int_{\tau}^{0} g_j(t, \vartheta)\, v_i(t, \vartheta)\, d\vartheta \qquad (j = 1, \ldots, n;\ i = 1, \ldots, n) \tag{4.26}$$

We shall examine the $i$th system and we shall look for the solution $v_i^{\circ}$ of equations (4.26) which satisfies the condition

$$\max\,(|\, v_i^{\circ}(t, \vartheta)\,| \text{ for } -\tau \leqslant \vartheta \leqslant 0) = \min \tag{4.27}$$

The solution of the problem (4.26) and (4.27) exists for every $i$ when conditions (3.3, $\tau$) are satisfied (and is therefore of the discontinuous type [14]). We shall assume, however, that the matrix $G[t, \tau, \vartheta]$ satisfies condition (3.3, $t$) in its strong sense (see p. 979). For this, it is sufficient that condition (3.23) of Definition 3.4 be satisfied for all $t^* \in [0, \infty]$. Then the solution $v_i^{\circ}(t, \vartheta)$ of the problem (4.26) and (4.27) is unique and is defined by the equality

$$v_i^{\circ}(t, \vartheta) = \alpha \operatorname{sign}\Big(\sum_{j=1}^{n} \lambda_i^{\circ} g_j(t, \vartheta)\Big) \qquad (-\tau \leqslant \vartheta \leqslant 0)$$

where $\alpha$, $\lambda_j^{\circ}$ are solutions of the problem

$$\frac{1}{\alpha} = \min_\lambda \Big\{ \int_{-\tau}^{0} \Big| \sum_{i=1}^{n} \lambda_j g_j(t, \vartheta) \Big|\, d\vartheta \Big\} \quad \text{for} \sum_{j=1}^{n} \lambda_j \delta_{ij} = \lambda_i = +1$$

Substituting into (4.24) the discontinuous function $V$ which was found, we shall find an expression for $K$, which in many cases can be simpler than in the case of the continuous function $V$ which was considered earlier.

We shall now seek the function $V^{[1]}$ in the form of a matrix with impulse elements

$$v_i(t,\vartheta)=\sum_{k=1}^{m}\alpha_k{}^i(t)\,\delta(\vartheta-\tau_k{}^i(t))$$

where $\delta$ represents the $\delta$-function. For that purpose, we shall seek [21] the elements $v_i(t,\vartheta)$ satisfying equations (4.26) in the form

$$\delta_{ij}=\int_{-\tau}^{0} g_j(t,\vartheta)\,d\eta_i(t,\vartheta)\qquad (d\eta_i(t,\vartheta)=v_i(t,\vartheta)\,d\vartheta) \tag{4.28}$$

where $d\eta_i$ is the measure of Stieltjes; we shall also require

$$\int_{-\tau}^{0} |\,d\eta_i(t,\vartheta)\,| = \min \tag{4.29}$$

The solution $v_i{}^\circ$ of the problem (4.28) and (4.29) is determined from the condition

$$v_i{}^\circ(t,\vartheta)=\sum_{k=1}^{m}\alpha_k{}^i(t)\,\delta(\vartheta-\tau_k{}^i(t)) \tag{4.30}$$

where $\alpha=\Sigma\,|\,\alpha_k{}^i\,|$ is the quantity

$$\alpha^{-1}=\min_\lambda\Big[\max_\vartheta\Big(\Big|\sum_{j=1}^{n}\lambda_j g_j(t,\vartheta)\Big|\Big),\ -\tau\leqslant\vartheta\leqslant 0\Big)\Big] \tag{4.31}$$
$$\text{for } \Sigma\,(\lambda_j\ \delta_{ij})=\lambda_i=+1$$

and $\tau_k{}^i$ are the points $\vartheta\in[-\tau,\ 0]$, where the quantity

$$\Big|\sum_{j=1}\lambda_j{}^\circ g_j(t,\vartheta)\Big|$$

reaches its maximum ($\lambda_j{}^\circ$ is the solution of the problem (4.31) and the number $m$ may depend on $t$ and $i$).

Let the impulse matrix $V(t,\vartheta)$ be found by the described procedure. Then the first component of the operator (4.18) takes the form of the vector

$$\Big\{\sum_{k=1}^{m(i)}\alpha_k{}^i(t)\,y\,(t-\tau_k{}^i(t))\Big\}\qquad (i=1,\ldots,n) \tag{4.32}$$

and this can be useful for the synthesis of the system. The second component, determined from (4.24), also appears in many cases to be more convenient than in the cases described earlier.

It is now possible to carry out the construction of equation (4.1) and verify simultaneously the validity of Theorem 4.1. For this purpose it is first of all necessary to substitute into equation (4.15) the optimal control $\zeta^{\circ}$ (4.9) found in the solution of Problem 4.1. We shall obtain a system of the form

$$dx / dt = P(t)x(t) + B(t)u(t), \qquad du / dt = P^{(1)}(t)x(t) + B^{(1)}(t)u(t) \tag{4.33}$$

asymptotically stable, satisfying conditions (4.16) and minimizing the integral $J_2$ (4.4). Then, in the second group of equations (4.33), $x(t)$ should be expressed as a function of $y(t+\vartheta)$ and $u(t+\vartheta)$ $(-\tau \leqslant \vartheta \leqslant 0)$ with the use of the operator $Y_{(1)}$ (4.17) and (4.18). Then we shall obtain the control law in the form of equation (4.1) in which the functions $A(t)$, $N(t, \vartheta)$, $M(t, \vartheta)$ are expressed in a known manner as functions of $P^{(1)}(t)$, $B^{(1)}(t)$, $Y(t, \vartheta)$, $K(t, \vartheta)$. The properties of the functions $A$, $N$, $M$, which are verified by Theorem 4.1 follow from the properties of $P^{(1)}$, $B^{(1)}$, $Y(t, \vartheta)$, $K(t, \vartheta)$, established earlier. Now, in order to conclude the proof of Theorem 4.1 it is sufficient to note that the closed system

$$\frac{dx}{dt} = P(t)x(t) + B(t)u(t) \tag{4.34}$$

$$\frac{du}{dt} = A(t)u(t) + \int_{-\tau}^{0} \{N(t, \vartheta)\, Q[t+\vartheta]\, x(t+\vartheta) + M(t, \vartheta)\, u(t+\vartheta)\}\, d\vartheta$$

satisfies condition (4.2) and the requirement of Problem 2.2 concerning the minimum of the integral $J$ (2.9), since system (4.33) has analogous properties; whatever the initial conditions $x_0(t_0+\vartheta)$, $u_0(t_0+\vartheta)$ $(-\tau \leqslant \vartheta \leqslant 0)$ of the system might be, its motion will coincide with some motion of the system beginning at the instant $t = t_0 + \tau$.

This follows directly from the method of construction of equation (4.1) which was described. The validity of Theorem 4.1 is proved.

**5. Solution of Problems 1.1 and 1.2.** We shall examine first Problem 1.1 and we shall formulate a theorem on the solution of this problem on the basis of the solution given in Section 4 by its first order approximation.

*Theorem 5.1.* If conditions (3.2, $\tau$) and (3.4, $\tau$) are satisfied by equation (2.4), (2.5) and (3.1) of the linear approximation of equations (1.3) and (1.4), Problem 1.1 has a solution which can be obtained from the solution (4.1) of this problem in the case of the linear approximation 2.1.

In fact, according to Theorem 4.1, the system (4.34) which can be constructed for conditions (3.2, τ) and (3.4, τ) satisfies condition (4.2) and has uniformly bounded coefficients. The system

$$\frac{dx}{dt} = p\,[t,\,x\,(t),\,u\,(t)] \tag{5.1}$$

$$\frac{du}{dt} = A\,(t)\,u\,(t) + \int_{-\tau}^{0} \{N\,(t,\,\vartheta)\,y\,(t+\vartheta) + M\,(t,\vartheta)\,u\,(t+\vartheta)\}\,d\vartheta \tag{5.2}$$

in which the vector $y(t)$ is determined by equation (1.4) differs from system (4.34) only by terms whose order of smallness in $x(t+\vartheta)$ and $u(t+\vartheta)$ is uniformly greater than the first (see (2.3)). According to Lemma 33.1 [5, p.191], system (4.34) admits a functional $v(t,\ x(\vartheta),\ u(\vartheta))$ which satisfies definitions of the type (33.4) to (33.6) [5,p.192]. This functional retains its properties of a Liapunov function in the neighborhood of the point $x = 0$, $u = 0$ also for system (5.1) and (5.2) as a consequence of condition (2.3). It follows that the motion $x = 0$, $u = 0$ is asymptotically stable on the basis of the equations of perturbed motion (5.1) and (5.2). Thus Problem 1.1 is solved and Theorem 5.1 proved.

We shall examine now Problem 1.2. We shall assume that in the neighborhood of the point $x = 0$, $u = 0$ the vector functions $p[t,\ x,\ u]$, $q[t,\ x]$ and $\omega[t,\ x,\ u,\ \zeta]$ from equations (1.3) and (1.4) and from the functional (1.8) can be expanded in power series with continuous coefficients uniformly bounded with respect to $t$.

Then, under the conditions of solvability of Problem 4.1 in the form of equations (4.15) and (4.9), which guarantee the fulfillment of condition (4.16), there exists an equation which can be constructed as a converging power series in $x$ and $u$

$$\frac{du}{dt} = A\,[t,\,x\,(t),u\,(t)] = P^{(1)}\,(t)\,x\,(t) + B^{(1)}\,(t)\,u\,(t) + $$
$$+ \sum p_k{}^j\,(t)\,x_1{}^{j_1}\,(t)\,\ldots\,x_n{}^{j_n}\,(t)\,u_1{}^{k_1}\,(t)\,\ldots\,u_r{}^{k_r}\,(t) \tag{5.3}$$

$(k+j=2,\ldots;\ j_1+\ldots+j_n=j;\ k_1+\ldots+k_r=k;\ p_k^j\,(t)$ **is an $r$-dimensional vector function)**

which guarantees the asymptotic stability of the motion $x = 0$, $u = 0$ and the minimum of the functional (1.8) on the basis of the system of equations (1.3) and (5.3) of the perturbed motion.

This result, concerning optimal nonlinear stabilization, and which we shall use here, was established first in [8] for the stationary case and extended afterwards [9] to the general nonstationary case. The problem was solved by the method of Liapunov functions with the use of some ideas of dynamic programming. V.I. Zubov informed the author that he has obtained an identical result for the nonstationary case by using the classical calculus of variations.

In order to solve Problem 1.2, one more result related to the prediction problem for linear systems will be necessary. A detailed analysis of this last problem is beyond the scope of the present paper and will be studied in a separate work. We shall give here only the necessary result for further use.

We shall examine the nonlinear problem, analogous to Problem 4.2 of linear prediction.

*Problem 5.1.* Find the operator $Y[t, y(\vartheta), u(\vartheta)]$ defined for $t \geqslant \tau$ for the continuous vector functions $\{y_j(\vartheta)\}$ $(j = 1, \ldots, l)$, $\{u_s(\vartheta)\}$ $(s = 1, \ldots, r)$ which lie in a sufficiently small neighborhood of the point $x = 0$, $u = 0$ and satisfying for the solutions $x(t)$, $u(t)$ of equation (1.3) (which lie in that neighborhood) the condition

$$x(t) = Y[t, y(t + \vartheta), u(t + \vartheta)] \tag{5.4}$$

where the vector $y(t)$ is related to $x(t)$ by equation (1.4).

The following statement is valid.

*Lemma 5.1.* Let system (3.1) and (2.5) of the first order approximation of equations (1.3) and (1.4) satisfy condition (3.4, $\tau$). Then Problem 5.1 has a solution. The operator $Y$ which is sought can be constructed as a series

$$Y[t, y(\vartheta), u(\vartheta)] = \sum_{k=1}^{\infty} Y_{(k)}[t, y(\vartheta), u(\vartheta)] \tag{5.5}$$

The first term of the series (5.5) can be chosen identical to the operator $Y_{(1)}$, which is constructed for Problem 4.2 according to Lemma 4.2. The other terms of the series (5.5) will then be determined from the systematic solution of the system of linear algebraic equations where the $k$th term of the series has in the neighborhood of the point $x = 0$, $u = 0$ the $k$th order in $x$ and $u$.

The following statement follows from the results which were found.

*Theorem 5.2.* Let equations (2.4), (2.5) and (3.1) of the linear approximation of equations (1.3) and (1.4) satisfy conditions (3.2, $\tau$) and (3.4, $\tau$). Then Problem 1.2 has as a solution in the form of the equation

$$du / dt = U [t, x (t + \vartheta), u (t + \vartheta)] \tag{5.6}$$

the right-hand side of which is constructed in the form of a series which converges in a small enough neighborhood of the point $x = 0$, $u = 0$. The linear approximation of equation (5.6) can be chosen in the form of equation (4.1) which solves Problem 1.2 in a first order approximation.

In order to be convinced of the validity of Theorem 5.2, it is sufficient to choose as equation (5.6) the equation

$$du / dt = A [t, Y [t, y (t + \vartheta), u (t + \vartheta)], u (t)] \tag{5.7}$$

in which the vector function $A$ is determined by the right-hand side of equation (5.3), the operator $Y$ is determined by condition (5.4) and the vector $y(t)$ is related to the vector $x(t)$ by equation (1.4).

We shall conclude here the discussion of the possibility of a solution in the nonstationary case of Problems 1.1 and 1.2 on the basis of their first order approximation.

**6. Solution of Problem 1.1 in the stationary case.** The solution of Problems 1.1 and 1.2 in the stationary case, i.e. in the case in which the functions $p$, $q$ and $\omega$ of (1.3), (1.4) and (1.8) or their first order approximations $P$, $B$, $Q$, $\omega_{(1)}$ do not depend directly on time $t$, are obtained naturally as consequences of Theorems 5.1 and 5.2 formulated in Section 5 for the general case. However, it is possible to give here some sufficient conditions of solvability of problems more general than those which follow directly from 5.1 and 5.2. We shall consider first the direct consequences of Theorems 5.1 and 5.2.

Let the matrices $P$, $B$, $Q$ and the quadratic form $\omega_{(1)}$ have constant coefficients $p_{ij}$, $b_{ij}$, $q_{ij}$, $c_{ij}$, $d_{ij}$, $e_{ij}$.

*Definition 6.1.* Equation (2.4) satisfies condition (6.1, 0) if there exists a set of numbers $j_k$ ($k = 1, \ldots, n$; $1 \leqslant j_k \leqslant n \times r$) for which the vectors $l[j_k]$ are linearly independent.

Here the vectors $l[j]$ are columns of the matrix (3.14) whereby the matrices $L_i$ in the stationary case have the form

$$L_i = P^{i-1}B \tag{6.1}$$

*Definition 6.2.* Equations (3.1) and (2.5) satisfy equation (6.2, 0) if there exists a set of numbers $j_k$($k = 1, \ldots, n$; $1 \leqslant j_k \leqslant n \times l$), for which the vectors $r[j_k]$ are linearly independent.

Here the vectors $r[j]$ are rows of the matrix (3.22), whereby the matrices $R_i$ in the stationary case have the form

$$R_i = QP^{i-1} \tag{6.2}$$

*Consequence 6.1.* If the stationary equations (2.4), (2.5) and (3.1) of the linear approximation for equations (1.3) and (1.4) satisfy conditions (6.1, 0) and (6.2, 0) Problem 1.1 has a solution. Equation (4.1) which determines the control law in the linear approximation, can be chosen stationary. If the functions $p$ and $q$ of equations (1.3) and (1.4) do not depend directly on time, then the nonlinear control law (5.2) can also be chosen stationary.

*Consequence 6.2.* Let us assume that the stationary equations (2.4), (2.5) and (3.1) satisfy the conditions (6.1, 0) and (6.2, 0). Then Problem 1.2 has a solution. If the function $\omega_{(1)}$ does not depend directly on time either, the linear approximation (4.1) of the optimal control law is also stationary. If the functions $p$, $q$ and $\omega$ in (1.3), (1.4) and (1.8) do not depend directly on time, the nonlinear control law (5.6) is stationary.

We shall now examine Problem 1.1 in the stationary case, and we shall formulate for its solvability a criterion which will take into account the known structure of the solutions in the stationary case of equations (3.1), in a manner similar to the one used for problems of analytical design of control system in [6].

We shall denote by the symbol $L[j_1, \ldots, j_m]$ $(m \leqslant r)$ the linear sub-space of the $n$-dimensional vectors $l[j_k + s]$ $(k = 1, \ldots, m;\ 1 \leqslant j_k \leqslant r;\ s = 0, r, 2r, \ldots, (n-1)r)$. The symbol $R[1, \ldots, n]$ will denote the linear sub-space generated by the vectors $r[j]$ $(j = 1, \ldots, n \times l)$, i.e. $R[1, \ldots, n]$ is a sub-space generated by all the row vectors $r[j]$ of the matrix $R$ (3.22). We shall denote by the symbol $K_{+0}$ some direct complement to the $n$-dimensional vector space of the original sub-space $K_-$ of the matrix $P$ generated by those of its eigenvalues which have negative real parts.

In the case of a simple structure of $P$, the sub-space $K_{+0}$ is a direct complement of the sub-space $K_-$ generated by the eigenvectors of the matrix $P$, which correspond to the roots $\rho_i$ with negative real parts of its characteristic equation

$$|P - \rho E|_1^n = 0 \tag{6.3}$$

In the general case, $K_{+0}$ is a direct complement of the sub-space $K_-$ of the initial conditions $x_0$, which generate those trajectories $x(x_0, t)$ of equation

$$dx / dt = Px \tag{6.4}$$

which converge towards the point $x = 0$ for $t \to \infty$.

The following statement is valid.

*Theorem 6.1.* Problem 2.1 can be solved if a set of numbers $j_1, \ldots, j_m$ can be found such that the enclosure

$$K_{+0} \subset L\,[j_1, \ldots, j_m] \subset R\,[1, \ldots, n] \tag{6.5}$$

is valid.

The control law, which stabilizes the object (2.4), can be chosen in the form

$$\frac{du}{dt} = Au(t) + \int_{-\tau}^{0} \{N(\vartheta)\, y\,(t+\vartheta) + M(\vartheta)\, u\,(t+\vartheta)\}\, d\vartheta \tag{6.6}$$

for which $A$ is a constant matrix, the matrices $N$ and $M$ are continuous and $\tau$ is an arbitrary chosen positive number.

We shall give the proof of the theorem. It is then sufficient to examine the case for which $L[j_1, \ldots, j_m]$ does not coincide with the whole space $x_i$ ($i = 1, \ldots, n$) because in the opposite case Theorem 6.1 follows directly from Consequence 6.1. Now let us assume that the dimension of $L[j_1, \ldots, j_m]$ is smaller than $n$. The proof of Theorem 6.1 is made in a similar manner to the proof of Theorem 4.1 with only some singularities which we shall mention here only briefly. First of all we shall make the linear transformation

$$x = Dx_* \tag{6.7}$$

which yields the coordinates $x_*^{[1]}$ controlled by the actions $u_{jk}$ ($k = 1, \ldots, m$). (A detailed analysis of the transformation (6.7) exceeds the frame of this paper.) The transformation (6.7) is chosen such that in the new coordinates $x_* = \{x_*^{[1]}, x_*^{[2]}\}$ equation (2.4) is decomposed into the system

$$\frac{dx_*^{[1]}}{dt} = P_{(1)}x_*^{[1]} + P_{(2)}x_*^{[2]} + B_{(1)}u_*^{[1]}, \qquad \frac{dx_*^{[2]}}{dt} = P_{(3)}x_*^{[2]} \qquad (u_*^{[2]} = 0) \tag{6.8}$$

where $x_*^{[1]}$ is an $s$-dimensional vector, $u_*^{[1]}$ an $m$-dimensional vector $\{u_k\}$, $x_*^{[2]}$ is an $(n - s)$-dimensional vector, the matrix $P_{(3)}$ has only eigenvalues with negative real parts, and equation

$$dx_*^{[1]} / dt = P_{(1)} x_*^{[1]} + B_{(1)} u_*^{[1]} \tag{6.9}$$

satisfies condition (6.1, 0). The decomposition of (2.4) into (6.8) is possible because of the left enclosure (6.5). An analogous transformation is made in [6]. (We shall note that in [6] there is an inaccuracy, pointed out by Zubov: in the transformation of the form (6.7) some terms are omitted. This inaccuracy in the proof in [6] can, however, be corrected.) As a consequence of (6.1, 0) system (6.9) can be stabilized by the control

$$du_*^{[1]} / dt = P_{(1)}{}^{\circ} x_*^{[1]}(t) + B_{(1)}{}^{\circ} u_*^{[1]}(t) \tag{6.10}$$

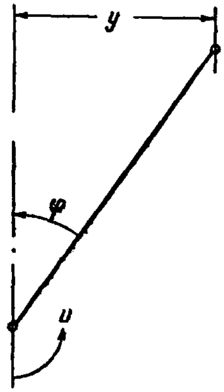

Fig. 1.

solved with the auxiliary problem 4.1. On the other hand, it can be verified that the right enclosure (6.5) guarantees the fulfillment for equations (2.5) and (3.1) of conditions for which it is possible to find an operator $Y_*[y(\vartheta), u(\vartheta)]$ satisfying the condition

$$x_*^{[1]}(t) = Y_*[y(t+\vartheta), u(t+\vartheta)] =$$
$$= \int_{-\tau}^{0} \{Y_*(\vartheta)\, y(t+\vartheta) + K_*(\vartheta)\, u(t+\vartheta)\}\, d\vartheta \qquad (t \geqslant \tau)$$

where $y(t)$ is related to $x(t)$ by (2.5), $x(t)$ and $u(t)$ are the displacements of the system (2.4), and $\tau$ is an arbitrarily chosen positive number. It is now possible to verify that the system

$$\frac{dx_*^{[1]}}{dt} = P_{(1)} x_*^{[1]}(t) + P_{(2)} x_*^{[2]}(t) + B_{(1)} u_*^{[1]}(t) + B_{(2)} u_*^{[2]}(t)$$
$$\frac{dx_*^{[2]}}{dt} = P_{(3)} x_*^{[2]}(t) + B_{(3)} u_*^{[2]}(t) \tag{6.11}$$
$$\frac{du_*^{[1]}}{dt} = B_{(1)}{}^{\circ} u_*^{[1]}(t) + \int_{-\tau}^{0} \{P_{(1)}{}^{\circ} [Y_*(\vartheta)\, y(t+\vartheta) + K_* u(t+\vartheta)]\}\, d\vartheta, \quad \frac{du_*^{[2]}}{dt} = -u_*^{[2]}$$

where $u_*^{[2]}$, the complement of $u_*^{[1]}$ up to the vector $u_*$, satisfies all the conditions of Theorem 6.1, which completes the proof of this theorem.

From Theorem (6.1) follows the statement.

*Theorem* 6.2. If the stationary equation of the first order approximation (2.4) and (2.5) satisfies conditions (6.5), Problem 1.1 has a solution; furthermore, the stabilizing control law can be chosen in the form of equation (6.6)

**7. Example.** We shall examine a simple illustrative example. Let us assume that it is required to stabilize a pendulum in its high, unstable equilibrium position by means of a moment $u(t)$ applied on its axis. It is furthermore possible to measure only the deviation $y(t)$ of the pendulum from the vertical, but neither the derivative $\dot{y}(t)$ nor the angular velocity of the oscillations of the pendulum (Fig. 1).

Let $\varphi = x_1$, $\dot{\varphi} = x_2$. Then by normalizing, if necessary, in an appropriate manner the scales of time, coordinates and forces, we shall write the equation of the perturbed motion of the considered object in the form

$$dx_1/\,dt = x_2, \qquad dx_2\,/\,dt = \sin x_1 + u \tag{7.1}$$

We shall choose equation (1.4) of feedback signal in the form

$$y\,(t) = \sin x_1\,(t) \tag{7.2}$$

The problem consists in the choice of a control law (5.2)

$$du\,/\,dt = au\,(t) + \int_{-\tau}^{0} (n\,(\vartheta)\,y\,(t+\vartheta) + m\,(\vartheta)\,u\,(t+\vartheta))\,d\vartheta \tag{7.3}$$

for which the unperturbed motion $x_1 = x_2 = u = 0$ is asymptotically stable on the basis of the equations of the perturbed motion (7.1),(7.2) and (7.3).

Equations (2.4) and (2.5) of the first order approximation for (7.1) and (7.2) have the form

$$dx_1\,/\,dt = x_2, \qquad dx_2\,/\,dt = x_1 + u, \qquad y = x_1 \tag{7.4}$$

We shall verify the fulfillment of conditions (6.1, 0) and (6.2, 0). The vectors $l[1]$ and $l[2]$ have the form

$$\begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \qquad \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}$$

and, consequently, these vectors are linearly independent. Condition (6.1, 0) is satisfied (the form (3.16) has here the form $\lambda_1^2 + \lambda_2^2$). The vectors $r[1]$ and $r[2]$ have the form

$$(1,\,0), \qquad (1,\,0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (0,\,1)$$

and are also independent. Condition (6.2, 0) is satisfied (the form

(3.23) is of the type $\lambda_1^2 + \lambda_2^2$). Thus Problem 4.1) can be solved. We shall choose the form $\omega_{(1)}$ as

$$\omega_{(1)} = x_1^2 + x_2^2 + u^2 + \zeta^2$$

and we shall seek the control $\zeta^0[x_1, x_2, u]$ minimizing the integral

$$I_2 = \int_0^\infty (x_1^2 + x_2^2 + u^2 + \zeta^2)\, dt$$

for the motions of system (7.4) and

$$du / dt = \zeta$$

We shall seek an optimal Liapunov function $v^0$ (4.5) and (4.7) in the form

$$v^0 = \alpha_{11}x_1^2 + \alpha_{22}x_2^2 + \alpha_{33}u^2 + 2\alpha_{12}x_1x_2 + 2\alpha_{13}x_1u + 2\alpha_{23}x_2u \tag{7.5}$$

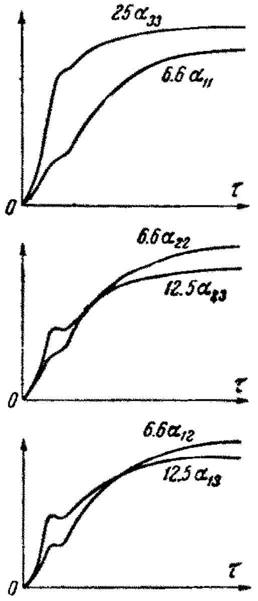


Fig. 2.

where $\alpha_{ij}$ = const. Equations (4.8) take here the form (by changing $dt$ into $-dt$, which is convenient in order to use method [29]) for the calculations (see earlier p.984)

$$\begin{aligned} d\alpha_{11} / dt &= 2\alpha_{12} - \alpha_{13}^2 + 1, & d\alpha_{12} / dt &= \alpha_{11} + \alpha_{22} - \alpha_{13}\alpha_{23} \\ d\alpha_{22} / dt &= 2\alpha_{12} - \alpha_{23}^2 + 1, & d\alpha_{13} / dt &= \alpha_{23} + \alpha_{12} - \alpha_{33}\alpha_{13} \\ d\alpha_{33} / dt &= 2\alpha_{23} - \alpha_{33}^2 + 1, & d\alpha_{23} / dt &= \alpha_{13} + \alpha_{22} - \alpha_{33}\alpha_{23} \end{aligned} \tag{7.6}$$

It is necessary to find a point of rest $\{\alpha_{ij}^0\}$ of equations (7.6) (where $d\alpha_{ij}/dt = 0$) for which the form $v^0$ (for $\alpha_{ij} = \alpha_{ij}^0$) is positive-definite. In the present case it is possible to solve directly the system of equations obtained from (7.6) for $d\alpha_{ij}/dt = 0$. We shall give, however the results of the calculations made according to method [29] which is valid for more complicated systems of higher order. An approximate solution of the problem on the computer MN-7 gave the following values:

$$\alpha_{11} = 11.17,\ \alpha_{22} = 10.05,\ \alpha_{33} = 3.21,\ \alpha_{12} = 9.88,\ \alpha_{13} = 4.60,\ \alpha_{23} = 4.60$$

(In Figs. 2, 3 and 4 are given the oscillograms of the corresponding trajectories of the auxiliary system of equations, having the same point of rest $\alpha_{ij}$ as (7.6.)

The calculation made according to method [29] on a digital computer gives a very accurate value of the quantities $\alpha_{ij}^{0}$.

Diagrams of the transient responses for equations (7.6), calculated on the computer "Ural 1" are given in Fig. 5. The values $\alpha_{ij}(t)$ giving $\alpha_{ij}^{0}$ for $t \to \infty$ are here the following:

$$\alpha_{11} = 11.1333433, \quad \alpha_{12} = 10.1333433, \quad \alpha_{22} = 10.1333433$$
$$\alpha_{13} = 4.6115817, \quad \alpha_{23} = 4.6115817, \quad \alpha_{33} = 3.1973682$$

This is by far superior to the accuracy required in practice.

The sought control $\zeta^{0}$ has here the form

$$\zeta^{\circ} = -(\alpha_{13}x_1 + \alpha_{23}x_2 + \alpha_{33}u) = -(4.6116x_1 + 4.6116x_2 + 3.1974u) \tag{7.7}$$

Consequently, the equation which solves Problem 4.1 has the form

$$\frac{du}{dt} = -4.6116x_1 - 4.6116x_2 - 3.1974u \tag{7.8}$$

It is now necessary to consider Problem 4.2. Since the coordinate $x_1(t)$ is known in the linear approximation on account of (7.4) and since its value can be fed into the control, there is no purpose apparently for introducing it into the control law (7.3) by means of $y(t+\vartheta)$. We shall assume that the quantity $\dot{x}_1(t)$ enters directly the element which forms $u(t)$. This will only add a term $bx_1(t)$ in equation (7.3). Then it becomes sufficient to represent only the quantity $x_2(t)$ by $y(t+\vartheta)$ $(-\tau \leqslant \vartheta \leqslant 0)$, i.e. it is necessary to seek in Problem 4.2) an operator $Y_{(1)}$ for which

$$x_2(t) = Y_{(1)}[y(t+\vartheta), u(t+\vartheta)] = \int_{-\tau}^{0} \{Y(\vartheta)\, y(t+\vartheta) + K(\vartheta)\, u(t+\vartheta)\}\, d\vartheta \tag{7.9}$$

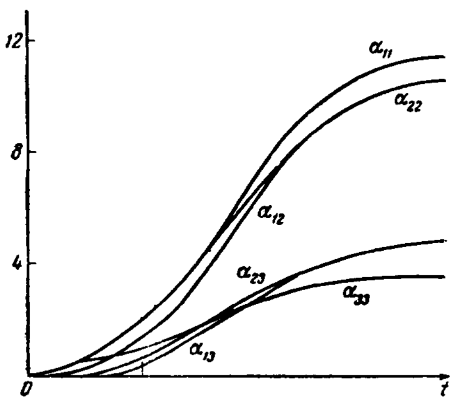


Fig. 3.

where $Y(\vartheta)$, $K(\vartheta)$ are scalar functions which do not depend directly on time, on account of the stationary nature of equation (7.4). The fundamental matrix $X[t, t+\vartheta]$ of the solutions of equations (3.1), i.e. in the present case of equation

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = x_1$$

has the form

$$X\,[t,\, t+\vartheta] = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix} \tag{7.10}$$

Let us choose $\tau = 1$. Then Problem 4.3 is transformed into the problem:

Find a function $V(\vartheta)$ $(-1 \leqslant \vartheta \leqslant 0)$ satisfying condition

$$x_{20} = \int_{-1}^{0} V(\vartheta)\,(x_{10}\cosh\vartheta + x_{20}\sinh\vartheta)\,d\vartheta$$

or

$$0 = \int_{-1}^{0} V(\vartheta)\cosh\vartheta\, d\vartheta, \qquad 1 = \int_{-1}^{0} V(\vartheta)\sinh\vartheta\, d\vartheta \tag{7.11}$$

According to (4.20) it is necessary to seek $V(\vartheta)$ in the form

$$V(\vartheta) = \lambda_1\cosh\vartheta + \lambda_2\sinh\vartheta \tag{7.12}$$

From (7.11) and (7.12) follow the equations for $\lambda_1$ and $\lambda_2$

$$\lambda_1 \int_{-1}^{0} \cosh^2(\vartheta)\, d\vartheta + \lambda_2 \int_{-1}^{0} \cosh\vartheta\sinh\vartheta\, d\vartheta = 0$$

$$\lambda_1 \int_{-1}^{0} \cosh(\vartheta)\sinh(\vartheta)\, d\vartheta + \lambda_2 \int_{-1}^{0} \sinh^2\vartheta\, d\vartheta = 1 \tag{7.13}$$

Hence $\lambda_1 = 7.358$, $\lambda_2 = 4.329$. Then, in accordance with (4.24), we conclude that the operator (7.9) is determined from the functions

$$Y(\vartheta) = 7.358\cosh\vartheta + 4.329\sinh\vartheta \tag{7.14}$$

$$K(\vartheta) = \int_{-1}^{\vartheta} \{(7.358\cosh\eta + 4.329\sinh\eta)\sinh(\eta - \vartheta)\}\, d\eta$$

By comparing (7.8), (7.9) and (7.14) we come to the conclusion that the stabilizing control law in linear approximation has the form

$$\frac{du}{dt} = -4.6116x_1(t) - 3.1974u(t) - 4.6116 \int_{-1}^{0} \Big\{(7.358\cosh\vartheta +$$

$$+ 4.329\sinh\vartheta)\, x_1(t+\vartheta) + \Big\{\int_{-1}^{\vartheta} \{7.358\cosh\eta + 4.329\sinh\eta\}\sinh(\eta - \vartheta)\, d\eta\, u(\vartheta)\Big\}\, d\vartheta$$

The closed nonlinear system will be described by equations (7.1), (7.2) and

$$\frac{du}{dt} = -4.6116\, y(t) - 3.1974 u(t) - 4.6116 \int_{-1}^{0} \Big\{ (7.358 \cosh\vartheta + 4.329 \sinh\vartheta)\, y(t+\vartheta) +$$

$$+ u(\vartheta) \int_{-1}^{\vartheta} \Big\{ (7.358 \cosh\eta + 4.329 \sinh\eta\} \sinh(\eta - \vartheta)\, d\eta) \Big\}\, d\vartheta \qquad (7.15)$$

We shall note that the stabilization of the system (7.1) by means of equation

$$\frac{du}{dt} = au(t) + bx_1(t) \qquad (7.16)$$

is not possible here, since the characteristic equation

$$\begin{vmatrix} -\rho & 1 & 0 \\ 1 & -\rho & 1 \\ b & 0 & a-\rho \end{vmatrix} = 0$$

of the system (7.4) and (7.16) for any choice of constants $a$ and $b$ has roots with a positive real part, and consequently the corresponding nonlinear system would also be unstable for any addition to (7.4) and (7.16) of nonlinear terms in $y(t)$ and $u(t)$. This substantiates here the necessity of introducing a delay into the control law (7.15), if it is assumed, as was done in the formulation of the problem, that it is not possible to measure directly the quantity $\dot{y}(t)$ (or $\phi(t)$) (on account of the presence of a disturbance of high frequency, for instance).

The author is grateful to V.E. Tret'iakov for making the calculations, oscillograms and diagrams given in this section, and to Iu.Sh.Gurevich for programming the computer "Ural 1".

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*Translated by* A.V.